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Multiquadric B-splines

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Our purpose in this paper is to show that the analogy between polynomial splines and generalized multiquadrics is very strong. In particular, combinations of multiquadrics, called ψ-splines are defined that are analogues of polynomial B-splines. This paper includes global linear independence, polynomial reproduction, and quasi-interpolation results for the span of the ψ-splines on non-uniform bi-infinite meshes which parallel those for polynomial B-splines. There are also results concerning the relationship between certain semi-infinite and bi-infinite combinations of ψ-splines. These results enable us to obtain error estimates for quasi-interpolation schemes involving multiquadrics based on a finite number of centres. © 1996 Academic Press, Inc.

1. INTRODUCTION

There has recently been a great deal of interest in approximation by radial basis functions, that is, by sums of translates of a single radially symmetric function. From this point of view univariate polynomial splines of odd degree are formed from polynomials plus sums of translates of the modulus raised to a fixed odd power. The generalized multiquadrics can then be viewed as obtained by smoothing out the derivative discontinuity of the function $|\cdot|^{2k-1}$. More precisely, let $k \in \mathbf{N}$ and $c > 0$. Then the basic generalized multiquadric of order $2k$ is defined by

$$\phi(x; 2k) = (x^2 + c^2)^{(2k-1)/2}, \tag{1.1}$$

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and the ϕ function centered at t_j by

$$\phi_{j,\,2k}(x) = \phi(x - t_j;\, 2k). \tag{1.2}$$

A multiquadric spline based on a finite number of centers is then a linear combination of appropriate $\phi_{j,\,2k}$'s supplemented by a polynomial of degree $2k-1$.

Our purpose in this paper is to show that the analogy between polynomial splines and generalized multiquadrics is very strong. In particular combinations of multiquadrics, called ψ-splines, will be defined that are analogues of polynomial B-splines. The paper includes global linear independence, polynomial reproduction, and quasi-interpolation results for the span of the ψ-splines on non-uniform bi-infinite meshes which parallel those for polynomial B-splines. Furthermore, it is shown that if a polynomial is expressed as a bi-infinite series of ψ-splines then corresponding semi-infinite series sum to half the polynomial plus a few generalized multiquadrics. This result allows us to obtain error estimates for quasi-interpolation by generalized multiquadrics based on a finite number of centers, from the results for quasi-interpolation by ψ-splines on non-uniform bi-infinite meshes. Our results extend those of Powell [7] and Beatson and Powell [2] for the case $k=1$ to general $k \in \mathbf{N}$.

2. PRELIMINARIES

In this section we define the ψ-splines and obtain some identities involving them.

Consider a mesh $\mathbf{t} = \cdots < t_{j-1} < t_j < t_{j+1} < \cdots$, with $t_{\pm j} \to \pm\infty$ as $j \to \infty$. Define the ψ-spline $\psi_{j,\,2k}\ (= \psi_{j,\,2k,\,t})$ as the weighted divided difference

$$\psi_{j,\,2k}(x) = \frac{t_{j+2k} - t_j}{2}\,[x - t_j,\, x - t_{j+1},\, \ldots,\, x - t_{j+2k}]\,\phi(x;\, 2k). \tag{2.1}$$

Hence making use of the expression of a divided difference as a linear combination of the function values at the indicated points, we can rewrite (2.1) as

$$\psi_{j,\,2k}(\mathrm{x}) = \frac{t_{j+2k} - t_j}{2}\,[t_j,\, t_{j+1},\, \ldots,\, t_{j+2k}]_u\,\phi(x - u;\, 2k), \tag{2.2}$$

where the subscript u, which will often be omitted, indicates that the divided difference is taken with respect to the u variable. This combination of $\phi_{j,\,2k}, ..., \phi_{j+2k,\,2k}$ will turn out to have some critical properties in common with the B-spline $N_{j,\,2k,\,t}$. (As is usual $N_{j,\,2k,\,t}$ denotes the B-spline of order $2k$ supported on $[t_j, t_{j+2k}]$ and normalized so that the sum of all the B-splines of a fixed order is 1.)

Before proceeding we need to derive some properties of the functions $\phi(x; 2k)$ and their derivatives and integrals. Firstly

$$D^{2k}\phi(x; 2k) = [(2k-1)!!]^2 \frac{c^{2k}}{(x^2+c^2)^{(2k+1)/2}}, \qquad k \in \mathbf{N}, \tag{2.3}$$

where as usual

$$m!! = \prod_{\{j : j = m (\mathrm{mod}\ 2) \text{ and } 0 < j \leqslant m\}} j.$$

Equation (2.3) can be shown by induction, the induction step following from applying Leibnitz's rule to

$$D^{2k+2}\{(x^2+c^2)(x^2+c^2)^{(2k-1)/2}\}.$$

Proceeding from (2.3) a similar induction argument shows that

$$D^{2k-1}\phi(x; 2k) = \frac{p(x; 2k)}{(x^2+c^2)^{(2k-1)/2}}, \qquad k \in \mathbf{N} \tag{2.4}$$

where $p(x; 2k)$ is an odd polynomial in x defined by the recurrence

$$p(x; 2k+2) = \begin{cases} x, & k = 0, \\ (2k+1)[(2k-1)!!]^2\, c^{2k} x & \\ \qquad + (2k+1)\, 2k(x^2+c^2)\, p(x; 2k), & k \in \mathbf{N}. \end{cases} \tag{2.5}$$

It follows immediately from this recurrence that the power expansion of $p(x; 2k)$ has positive coefficients and that

$$p(x; 2k) = (2k-1)!\, x^{2k-1} + \sum_{j=1}^{k-1} a_{j,\,2k}\, c^{2k-2j} x^{2j-1}, \tag{2.6}$$

for some constants $a_{j,\,2k}$ not depending on c. Hence

$$D^{2k-1}\phi(x; 2k) = \pm(2k-1)! + \mathcal{O}(|x|^{-2}), \qquad \text{as} \quad x \to \pm\infty. \tag{2.7}$$

Defining

$$A(k) = \frac{(2k-1)!!}{(2k)!!} = \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6} \cdots \frac{(2k-1)}{2k} \tag{2.8}$$

we have that

$$\int_{-\infty}^{\infty} (x^2+c^2)^{-(2k+1)/2}\,dx = \frac{1}{kA(k)\,c^{2k}}, \qquad k \in \mathbf{N}. \tag{2.9}$$

This can be proven by induction, the induction step following from the easily verified identity

$$\int (x^2+c^2)^{-(2k+1)/2}\,dx = \frac{x}{(2k-1)\,c^2}\,(x^2+c^2)^{-(2k-1)/2} + \frac{2(k-1)}{(2k-1)\,c^2}\int (x^2+c^2)^{-(2k-1)/2}\,dx. \tag{2.10}$$

Also trivially

$$\int x(x^2+c^2)^{-(2k+1)/2}\,dx = -\frac{1}{2k-1}\,(x^2+c^2)^{-(2k-1)/2} + M, \tag{2.11}$$

so that

$$\int_{-\infty}^{\infty} |x|\,(x^2+c^2)^{-(2k+1)/2}\,dx = \frac{2}{(2k-1)\,c^{2k-1}}. \tag{2.12}$$

Recall now that a multiple of the polynomial B-spline is the Peano kernel of the divided difference, so that in particular

$$[t_j, ..., t_{j+2k}]\,g = \frac{2k}{(t_{j+2k}-t_j)}\,\frac{1}{(2k)!}\int_{-\infty}^{\infty} N_{j,\,2k,\,t}(u)\,g^{(2k)}(u)\,du. \tag{2.13}$$

Using this with $g(u) = \phi(x-u;\,2k)$ we find that

$$\psi_{j,\,2k}(x) = \frac{k}{(2k)!}\int_{-\infty}^{\infty} N_{j,\,2k,\,t}(u)\,\phi^{(2k)}(x-u)\,du. \tag{2.14}$$

Applying (2.3)

$$\psi_{j,\,2k}(x) = kA(k)\,c^{2k}\int_{-\infty}^{\infty} N_{j,\,2k,\,t}(u)((x-u)^2+c^2)^{-(2k+1)/2}\,du. \tag{2.15}$$

It follows from this and the formula

$$\int_{-\infty}^{\infty} N_{j,\,l,\,t}(x)\,dx = \frac{t_{j+l}-t_j}{l}, \tag{2.16}$$

for the integral of a B-spline that $\psi_{j,2k}$ is nonnegative and decays like $[t_{j+2k}-t_j]\, d(x, [t_j, t_{j+2k}])^{-(2k+1)}$ as $x \to \pm\infty$, where $d(\cdot, \cdot)$ denotes the usual distance for $\mathbf{R}$. Also

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \psi_{j,2k}(x) &= kA(k)c^{2k} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} N_{j,2k,t}(u)((x-u)^2+c^2)^{-(2k+1)/2}\, du,\\ &= kA(k)c^{2k} \int_{-\infty}^{\infty} ((x-u)^2+c^2)^{-(2k+1)/2}\, du,\\ &= 1, \end{aligned} \tag{2.17}$$

where in the last step we have used (2.9). Let

$$S_{2k}(u) = kA(k)\; c^{2k}(u^2+c^2)^{-(2k+1)/2}, \qquad k \in N.$$

Jones [6, p. 178] gives the formula

$$\int_{-\infty}^{\infty} e^{-itx}(1+x^2)^{-(k+1/2)}\, dx = \frac{\pi^{1/2}\, |t|^k\, K_k(|t|)}{(k-\frac{1}{2})!\; 2^{k-1}},$$

where K_k is a modified Bessel function. Hence the Fourier transform of S_{2k} is

$$\widehat{S_{2k}}(t) = kA(k)c^{2k} \frac{2\pi^{1/2}}{(k-\frac{1}{2})!} \left|\frac{t}{2c}\right|^k K_k(|ct|). \tag{2.18}$$

Now from Abromowitz and Stegun [1, p. 375]

$$\begin{aligned} K_k(z) = &\frac{1}{2}\left(\frac{1}{2}z\right)^{-k} \frac{(k-j-1)!}{j!}\left(-\frac{1}{4}z^2\right)^j\\ &+(-)^{k+1} \ln\left(\frac{1}{2}z\right) I_k(z)\\ &+(-)^k \frac{1}{2}\left(\frac{1}{2}z\right)^k \sum_{j=0}^{\infty} (\eta(j+1)+\eta(k+j+1)) \frac{(\frac{1}{4}z^2)^j}{j!(k+j)!} \end{aligned}$$

where here η denotes the digamma function, and

$$I_k(z) = \frac{(\frac{1}{2}z)^k}{\Gamma(k+1)} + \mathcal{O}(z^{k+2}), \qquad \text{as} \quad z \to 0.$$

Hence

$$\widehat{S_{2k}}(t)=\sum_{j=0}^{k-1}\frac{(k-j-1)!}{(k-1)!\,j!}\left(\frac{-c^2}{4}\right)^4 t^{2j}$$
$$+kA(k)c^{2k}\left\{(-)^{k+1}\frac{2\pi^{1/2}}{(k-\frac{1}{2})!}\left(\frac{t}{2}\right)^{2k}\ln\left(\frac{c\,|t|}{2}\right)(1+\mathcal{O}(c^2t^2))\right.$$
$$\left.+\mathcal{O}(t^{2k})\right\} \qquad \text{as} \quad t\to 0. \tag{2.19}$$

Since $K_k(z)$ is infinitely differentiable on $\mathbf{R}\setminus\{0\}$, it follows from (2.19) that $\widehat{S_{2k}}\in C^{2k-1}(\mathbf{R})$. Since $K_k(z)$ is positive for $z>0$ and $k>-1$, $\widehat{S_{2k}}(t)$ is positive for $t\in R$. Also from (2.19)

$$\widehat{S_{2k}}^{(2j)}(0)=\frac{(k-j-1)!}{(k-1)!\,j!}\left(\frac{-c^2}{4}\right)^j(2j)!, \qquad 0\leqslant j\leqslant k-1.$$

Hence from the formula $(\widehat{x^r f})(t)=(id/dt)^r\hat{f}(t)$

$$\int_{\mathbf{R}} x^{2j}S_{2k}(x)\,dx=\frac{(2j)!\,(k-j-1)!}{(k-1)!\,j!}\left(\frac{c^2}{4}\right)^j \tag{2.20}$$

for $0\leqslant j\leqslant k-1$. Applying Cauchy–Schwarz we have

$$\int_{\mathbf{R}}|x|^{2j-1}S_{2k}(x)\,dx\leqslant\left[\int_{\mathbf{R}}x^{2j}S_{2k}(x)\,dx\int_{\mathbf{R}}x^{2j-2}S_{2k}(x)\,dx\right]^{1/2}$$
$$=\mathcal{O}(c^{2j-1}), \qquad 0\leqslant j\leqslant k-1.$$

Also note that in the particular case $j=1$ (2.20) gives

$$\int_{\mathbf{R}}x^2S_{2k}(x)\,dx=\frac{c^2}{2(k-1)}, \qquad k\geqslant 2. \tag{2.21}$$

3. BASIC PROPERTIES OF ψ-SPLINES

In this section we derive some fundamental properties of the ψ-splines. These include global linear independence, polynomial reproduction properties, and expressions for ϕ and ψ splines as convolutions of a kernel with a power of the modulus and polynomial B-splines respectively.

It will be convenient to have the following notation. Given an infinite mesh $\mathbf{t}:\cdots<t_{j-1}<t_j<t_{j+1}<\cdots$ we define a coefficient sequence $\mathbf{d}=\{d_j\}_{j=-\infty}^{\infty}$ to be in the growth class $C(2k,\mathbf{t})$ if $d_j=\mathcal{O}(|t_j|^{2k-1})$ as $j\to\pm\infty$. We note that for meshes $\mathbf{t}$ of finite mesh size the condition is equivalent to the condition $\sum_{j=-\infty}^{\infty} d_j N_{j,2k}(x) = \mathcal{O}(|x|^{2k-1})$ as $x \to \pm\infty$. (See the proof of Lemma 3.)

We remind the reader of the following well known result.

Lemma 1 (Local linear independence of the B-spline basis). *Let $k \in \mathbf{N}$ and consider an infinite mesh $\mathbf{t}: \cdots < t_{j-1} < t_j < t_{j+1} < \cdots$. Let $s = \sum_{j=-\infty}^{\infty} a_j N_{j,k}$ and $r = \sum_{j=-\infty}^{\infty} b_j N_{j,k}$. Then $s(x) = r(x)$ for all $x \in (t_j, t_{j+1})$ if and only if $a_l = b_l$ for all $j - k < l \leqslant j$.*

Proof. Let p be any polynomial of degree $k-1$. Then p can be expressed as a linear combination of B-splines of order k. Because of the support properties of the B-splines this implies that $E := \{N_{l,k} : j-k < l \leqslant j\}$ is a spanning set for the polynomials of degree $k-1$ considered as a vector space of functions from (t_j, t_{j+1}) to $\mathbf{R}$. From the cardinality of E it follows that E is not merely a spanning set but also a basis for this vector space. This is the required result. ∎

The next result expresses ψ-splines as a convolution. It is a generalization of a result of Powell [7] in the case of the ordinary multiquadric.

Lemma 2 (ψ-splines as convolutions). *Let $k \in \mathbf{N}$ and consider an infinite mesh*

$$\mathbf{t}: \cdots < t_{j-1} < t_j < t_{j+1} < \cdots, \; t_{\pm j} \to \pm\infty, \qquad as \quad j \to \infty,$$

and $h = \sup_j (t_{j+1} - t_j) < \infty$. Suppose $\boldsymbol{\alpha} = \{\alpha_j\}_{j=-\infty}^{\infty} \in C(2k, \mathbf{t})$. Then

$$s(x) = \sum_{j=-\infty}^{\infty} \alpha_j \psi_{j,2k,t}(x)$$

is absolutely convergent for each x and is given by

$$s(x) = kA(k)c^{2k} \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \alpha_j N_{j,2k,t}(u)((x-u)^2 + c^2)^{-(2k+1)/2} \, du$$

in which the integral is absolutely convergent.

Proof. Define g as the *B*-spline series

$$g(x) = \sum_j \alpha_j N_{j,2k,t}(x)$$

and

$$M(x) = \sum_j |\alpha_j|\, N_{j,\,2k,\,t}(x).$$

As is familiar there is no convergence problem with these series as only a finite number of the B-splines are non-zero at any x. Indeed on $[t_i, t_{i+1}]$ only $N_{i-2k+1,\,2k,\,t}, \ldots, N_{i,\,2k,\,t}$ are non-zero. From this, the growth condition on $\boldsymbol{\alpha}$, the partition of unity property of the B-splines, and the finiteness of the mesh size, it follows that

$$|g(x)| \leqslant M(x) = \mathcal{O}(|x|^{2k-1}) \qquad \text{as} \quad x \to \pm\infty.$$

Hence, for each fixed, x

$$0 \leqslant kA(k)\, c^{2k} M(u)((x-u)^2 + c^2)^{-(2k+1)/2} = \mathcal{O}(u^{-2}), \qquad \text{as} \quad u \to \pm\infty.$$

Thus the middle quantity above is integrable with respect to u on $\mathbf{R}$. It now follows from (2.15) and the Lebesgue dominated convergence theorem that

$$\sum_{j=-\infty}^{\infty} \alpha_j \psi_{j,\,2k,\,t}(x) = kA(k) c^{2k} \int_{-\infty}^{\infty} \sum_j \alpha_j N_{j,\,2k,\,t}(u)((x-u)^2 + c^2)^{-(2k+1)/2}\, du$$

with the series on the left converging absolutely for each x. ∎

LEMMA 3. (Polynomials in the space spanned by the ψ-splines). *Let* $\mathbf{t}$ *satisfy the conditions of Lemma* 2. *Suppose that* $p \in \pi_{2k-1}$ *has B-spline series expansion*

$$p(x) = \sum_j d_j N_{j,\,2k}(x).$$

Then

$$s(x) = \sum_j d_j \psi_{j,\,2k}(x),$$

is a polynomial of the same degree as p and with the same leading coefficient. Moreover, if $p \in \pi_1$ *then s and p are identical and*

$$d_j = p(t_j^*), \qquad j = 0, \pm 1, \pm 2, \ldots,$$

where the points

$$t_j^* = \frac{t_{j+1} + \cdots + t_{j+2k-1}}{2k-1},$$

are the special points occurring in the definition of Schoenberg's variation diminishing spline.

Proof. Recall the remarkable condition property of the B-spline basis (see for example de Boor [5, p. 155])

$$|d_i| \leqslant D_{2k} \left\| \sum_j d_j N_{j,\,2k} \right\|_{L^\infty[t_{i+1},\, t_{i+2k-1}]}$$

where D_{2k} is independent of the mesh. Hence

$$|d_i| \leqslant D_{2k} \, \|p\|_{L^\infty[t_{i+1},\, t_{i+2k-1}]}.$$

This together with the finiteness of the mesh size, implies that the coefficients belong to the growth class $C(2k, \mathbf{t})$. Hence, from Lemma 2,

$$\begin{aligned} s(x) &= kA(k)\, c^{2k} \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} d_j N_{j,\,2k}(u)((x-u)^2+c^2)^{-(2k+1)/2}\, du \\ &= kA(k)\, c^{2k} \int_{-\infty}^{\infty} p(x-u)(u^2+c^2)^{-(2k+1)/2}\, du. \end{aligned} \tag{3.1}$$

Supposing now p is of exact degree m, $0 \leqslant m \leqslant 2k-1$, so that

$$p(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0,$$

with $a_m \neq 0$, (3.1) implies

$$\begin{aligned} s(x) &= kA(k)\, c^{2k} a_m x^m \int_{-\infty}^{\infty} (u^2+c^2)^{-(2k+1)/2}\, du \\ &\quad + \sum_{i=0}^{m-1} b_i x^i \int_{-\infty}^{\infty} p_i(u)(u^2+c^2)^{-(2k+1)/2}\, du, \end{aligned}$$

where $p_i(u) \in \pi_m$. The first part of the lemma follows.

For $p \in \pi_1$ Schoenberg's variation diminishing spline with coefficients $d_j = p(t_j^*)$ satisfies

$$p(x) = \sum_j p(t_j^*)\, N_{j,\,2k}(x), \ \forall x.$$

From Lemma 1, that is the local linear independence of the B-splines, the coefficients $p(t_j^*)$ are the only coefficients with this property. An application of (3.1) now gives

$$\begin{aligned} s(x) &= \sum_j p(t_j^*)\, \psi_{j,\,2k}(x) \\ &= kA(k)\, c^{2k} \int_{-\infty}^{\infty} p(x-u)(u^2+c^2)^{-(2k+1)/2}\, du \\ &= kA(k)\, c^{2k} p(x) \int_{-\infty}^{\infty} (u^2+c^2)^{-(2k+1)/2}\, du \\ &= p(x), \end{aligned}$$

where in the second to last step we have used that if g is odd and integrable $\int_{-\infty}^{\infty} g = 0$. ∎

Unfortunately when p has degree greater than 1 the coefficients used to express p in terms of the B-splines do not suffice to express it in terms of the ψ-splines. For example if $k > 1$ and

$$p(x) = x^2 = \sum_j d_j N_{j,\,2k}(x),$$

then by (2.21)

$$\begin{aligned} s(x) &= \sum_j d_j \psi_{j,\,2k}(x) \\ &= x^2 + kA(k)\, c^{2k} \int_{-\infty}^{\infty} u^2 (u^2+c^2)^{-(2k+1)/2}\, du \\ &= x^2 + Mc^2 \end{aligned}$$

where the non-zero constant M, depends on k.

LEMMA 4 (Global linear independence of the ψ-spline basis). *Let* $\mathbf{t}$ *be as in Lemma* 2 *and suppose* $\mathbf{d} \in C(2k, \mathbf{t})$. *Then*

$$s(x) = \sum_j d_j \psi_{j,\,2k}(x) = 0, \qquad \textit{for all } x,$$

implies $\mathbf{d}$ *is the zero sequence.*

Proof. The assumptions on $\mathbf{t}$ and $\mathbf{d}$ ensure

$$g = \sum_j d_j N_{j,\,2k},$$

satisfies $g(x) = \mathcal{O}(|x|^{2k-1})$ as $x \to \pm\infty$ hence is a tempered distribution, having a generalized Fourier Transform well defined except possibly at zero. (The properties of tempered distributions we use can be found in Rudin [8, particularly pp. 173–178].) Now

$$0 = s(x) = \sum_j d_j \psi_{j,2k}(x), \qquad \text{for all } x,$$

implies by Lemma 2

$$0 = kA(k)\, c^{2k} \left\{ \sum_j d_j N_{j,2k}(\cdot) \right\} * (\cdot^2 + c^2)^{-(2k+1/2)}\}, \qquad \text{for all } x. \quad (3.2)$$

Taking the generalized Fourier transform we obtain

$$0 = \hat{g}(\xi)\, \widehat{S_{2k}}(\xi), \qquad \text{for all} \quad \xi \neq 0, \quad (3.3)$$

where $\widehat{S_{2k}} \in C^{2k-1}(\mathbf{R})$ is the everywhere positive transform of S_{2k}, previously discussed (see (2.18) and (2.19)) . Hence (3.3) implies the support of $\hat{g}(\xi)$ is $\xi = 0$. Thus g must be a polynomial. Since from above $g(x) = O(|x|^{2k-1})$, this polynomial has exact degree not exceeding $2k-1$. Then, from Lemma 3, s is a polynomial of the same exact degree. But from the hypotheses s being identically zero has exact degree -1. Hence $\sum d_j N_{j,2k}$ is identically zero. The result follows from Lemma 1. ∎

The following corollary goes in the opposite direction to Lemma 3.

Corollary 5 (Polynomial reproduction and a dual representation). *Let* $k \in \mathbf{N}$, *and* $\mathbf{t}$ *be as in Lemma* 2. *Let*

$$s(x) = \sum_j d_j \psi_{j,2k}(x) \qquad \text{and} \qquad g(x) = \sum_j d_j N_{j,2k}(x). \quad (3.4)$$

where the first sum may be divergent. Then:

(a) *Given any polynomial* $q \in \pi_{2k-1}$ *there is a unique choice of coefficients* $\mathbf{d}$ *such that both* $s = q$ *and the growth condition* $\mathbf{d} \in C(2k, \mathbf{t})$ hold.

(b) *If* $\mathbf{d} \in C(2k, \mathbf{t})$ *and* $s \in \pi_{2k-1}$ *then* $g \in \pi_{2k-1}$ *and has the same exact degree and leading coefficient as* s.

(c) *If* $\mathbf{d} \in C(2k, \mathbf{t})$ *and* $s \in \pi_1$ *then* s *and* g *are identical.*

Proof. (a) Let $q \in \pi_{2k-1}$ be fixed and of exact degree m. From Lemma 3 choosing $p(x)$ there as $x^m, x^{m-1}, \ldots, 1$ in turn, and then taking linear combinations, we can find coefficients $\mathbf{d}$ satisfying the growth condition $\mathbf{d} \in C(2k, \mathbf{t})$ and $s = q$. From Lemma 4 these coefficients are unique. Note that in this construction $\sum_j d_j N_{j,2k}$ is a polynomial with the same exact degree and leading coefficient as q.

(b) Let $\mathbf{d} \in C(2k, \mathbf{t})$ and $s \in \pi_{2k-1}$. Then from the uniqueness part of part (a) the coefficients $\mathbf{d}$ must be those of the construction of part (a). The conclusion follows from the remark at the end of the proof of part (a).

(c) From (b) if $s \in \pi_1$ then $g \in \pi_1$. The conclusion then follows from Lemma 3.

LEMMA 6. *Let $k \in \mathbf{N}$ and $c > 0$. Then*

$$I_k = \int \frac{u^{2k-1}}{((x-u)^2 + c^2)^{(2k+1)/2}}\, du = \left\{ \frac{p(x,u)}{((x-u)^2 + c^2)^{(2k-1)/2}} \right\} + C, \tag{3.5}$$

where $p(x, u)$, considered as a polynomial in u, has degree $2k-1$ and constant part

$$-\frac{(x^2 + c^2)^{2k-1}}{2kA(k)\, c^{2k}}. \tag{3.6}$$

Proof. By differentiation one easily establishes the recurrences

$$\int \frac{u^m}{((x-u)^2 + c^2)^{(2k+1)/2}}\, du$$

$$= \frac{u^{m-1}}{(m-2k)((x-u)^2 + c^2)^{(2k-1)/2}}$$

$$- \left(\frac{2k - 2m + 1}{m - 2k} \right) x \int \frac{u^{m-1}}{((x-u)^2 + c^2)^{(2k+1)/2}}\, du$$

$$- \left(\frac{m-1}{m-2k} \right) (x^2 + c^2) \int \frac{u^{m-2}}{((x-u)^2 + c^2)^{(2k+1)/2}}\, du$$

and

$$\int ((x-u)^2 + c^2)^{-(2k+1)/2}\, du = \frac{(u-x)}{(2k-1)\, c^2} ((x-u)^2 + c^2)^{-(2k-1)/2}$$

$$+ \frac{2(k-1)}{(2k-1)\, c^2} \int ((x-u)^2 + c^2)^{-(2k-1)/2}\, du.$$

Since

$$\int ((x-u)^2 + c^2)^{-3/2}\, du = \frac{(u-x)}{c^2} ((x-u)^2 + c^2)^{-1/2} + D,$$

an easy induction shows that there is an indefinite integral I_k of the form stated in (3.5). It remains to show that the constant part of the polynomial in u, $p(x, u)$, is given by (3.6).

To this end make the substitution

$$\cos\theta = \frac{c}{\sqrt{(x-u)^2+c^2}} \qquad \text{and} \qquad \sin\theta = \frac{-(x-u)}{\sqrt{(x-u)^2+c^2}},$$

implying $u = x + c\tan\theta$. Note in particular that the expression defining $\cos\theta$ is defined everywhere and is always positive. Then

$$\begin{aligned} I_k &= \int \frac{(x+c\tan\theta)^{2k-1} c\sec^2\theta}{(c\sec\theta)^{2k+1}}\, d\theta \\ &= \frac{1}{c^{2k}} \int (x\cos\theta + c\sin\theta)^{2k-1}\, d\theta \\ &= \frac{(x^2+c^2)^{(2k-1)/2}}{c^{2k}} \int \cos^{2k-1} t\, dt \end{aligned}$$

where

$$t = \theta - \gamma, \qquad \cos\gamma = \frac{x}{\sqrt{x^2+c^2}} \qquad \text{and} \qquad \sin\gamma = \frac{c}{\sqrt{x^2+c^2}}.$$

Then with $v = \sin t$

$$\int \cos^{2k-1} t\, dt = \int (1-v^2)^{k-1}\, dv = \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j v^{2j+1}}{2j+1} + E.$$

Therefore we may choose

$$p(x, u) = \left(\frac{c}{\cos\theta}\right)^{2k-1} \frac{(x^2+c^2)^{(2k-1)/2}}{c^{2k}} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j \sin^{2j+1} t}{2j+1}.$$

When $u = 0$, $\cos\theta = c/\sqrt{x^2+c^2} = \sin\gamma$, $\sin\theta = -x/\sqrt{x^2+c^2} = -\cos\gamma$, and $\sin t = \sin(\theta-\gamma) = \cos\gamma\sin\theta - \cos\theta\sin\gamma = -1$. Therefore

$$p(x, 0) = -\frac{(x^2+c^2)^{2k-1}}{c^{2k}} \sum_{j=0}^{k-1} \frac{(-1)^j \binom{k-1}{j}}{2j+1}.$$

Finally

$$\frac{1}{2kA(k)}=\frac{(2k-2)!!}{(2k-1)!!}=\int_0^{\pi/2}\sin^{2k-1}x\,dx=\sum_{j=0}^{k-1}\frac{(-1)^j\binom{k-1}{j}}{2j+1},$$

which completes the proof. ∎

Lemma 7 (ϕ and ψ splines as convolutions). *Let $k\in\mathbf{N}$. Then*

$$\phi(x;2k)=kA(k)c^{2k}\,|\cdot|^{2k-1}*(\cdot^2+c^2)^{-(2k+1)/2}, \tag{3.7}$$

and

$$\psi_{j,2k}(x)=kA(k)\,c^{2k}N_{j,2k,t}*(\cdot^2+c^2)^{-(2k+1)/2}. \tag{3.8}$$

Proof. For $f\in C^{2k}(\mathbf{R})$ of compact support a straightforward integration by parts argument shows

$$f(x)=\frac{1}{2\cdot(2k-1)!}\,(|\cdot|^{2k-1}*f^{(2k)})(x).$$

That this also holds for the function ϕ, which grows at infinity, follows from the following more direct argument.

Let $g(x,u)$ be the indefinite integral

$$\int\frac{u^{2k-1}}{((x-u)^2+c^2)^{(2k+1)/2}}\,du=\frac{p(x,u)}{((x-u)^2+c^2)^{(2k-1)/2}}+C,$$

discussed in Lemma 6, with C chosen to be 0. Then

$$\int_{-\infty}^{\infty}\frac{|u|^{2k-1}}{((x-u)^2+c^2)^{(2k+1)/2}}\,du=\lim_{u\to\infty}g(x,u)+\lim_{u\to-\infty}g(x,u)-2g(x,0).$$

But from the previous lemma the first two terms on the right above cancel and the last term equals

$$\frac{(x^2+c^2)^{(2k-1)/2}}{kA(k)c^{2k}},$$

which establishes (3.7).

The second part of the lemma is already contained in (2.15) and Lemma 2. ∎

4. POLYNOMIALS AS SEMI-INFINITE SUMS OF ψ-SPLINES

Fundamental to the work of Beatson and Powell [2] is that linear polynomials are not only in the space of all bi-infinite combinations of $\psi_{j,2}$'s but also in the space of semi-infinite combinations (modulo a few *edge* ϕ's). In this section we will obtain an analogous result for ψ-splines of general order.

The proof used in [2] was a direct integration. An alternative collapsing sum argument is as follows. Let

$$\beta(x) = \sum_{j=0}^{\infty} \psi_{j,2k}(x).$$

Then

$$\begin{aligned}
\beta(x) &= \lim_{m\to\infty} \sum_{j=0}^{m} \psi_{j,2k}(x) \\
&= \tfrac{1}{2} \lim_{m\to\infty} \sum_{j=0}^{m} \{[t_{j+1}, ..., t_{j+2k}]\, \phi(x-u; 2k) \\
&\qquad - [t_j, ..., t_{j+2k-1}]\, \phi(x-u; 2k)\} \\
&= \tfrac{1}{2} \{\lim_{m\to\infty} [t_{m+1}, ..., t_{m+2k}]\, \phi(x-u; 2k) - [t_0, ..., t_{2k-1}]\, \phi(x-u; 2k)\}
\end{aligned}$$

where all the divided differences are with respect to the u variable. Using the asymptotic expression for $\phi^{(2k-1)}$ (2.7) to express the first term on the right, and the familiar formula

$$[t_l, t_{l+1}, ..., t_{l+n}] f = \sum_{j=l}^{l+n} \frac{f(t_j)}{\prod_{i=l,\, i\neq j}^{l+n} (t_j - t_i)}$$

for a divided difference to express the last, it follows that

$$\beta(x) = \frac{1}{2} - \frac{1}{2} \sum_{j=0}^{2k-1} \left[\prod_{i=0,\, i\neq j}^{2k-1} (t_j - t_i) \right]^{-1} \phi_{j,2k}(x).$$

More generally we have

THEOREM 8 (Polynomials as semi-finite sums of ψ-splines). *Suppose* $\mathbf{t}$ *satisfies the conditions of Lemma* 2 *and* $\mathbf{d} \in C(2k, \mathbf{t})$. *Further suppose* $p = \sum_{j=-\infty}^{\infty} d_j \psi_{j,2k}$ *is in* π_{2k-1}. *Then* $q = \sum_{j=-\infty}^{\infty} d_j N_{j,2k}$ *is also in* π_{2k-1}. *Furthermore the function s, defined by the semi-infinite sum*

$$s(x) = \sum_{j=0}^{\infty} d_j \psi_{j,2k}(x),$$

can be rewritten as

$$s(x)=\frac{p(x)}{2}+\frac{1}{2}\sum_{l=0}^{2k-1}\lambda_l\phi_{l,2k}(x),$$

where the vector λ *is the unique solution of*

$$\sum_{l=0}^{2k-1}\lambda_l(\cdot-t_l)^{2k-1}=q.$$

The theorem also holds in the polynomial spline case $c=0$.

Proof. We consider firstly the case when $c=0$ so that $\psi_{j,2k}$ is the polynomial B-spline $N_{j,2k}$ and p and q are identical. Then

$$s(x)=\sum_{j=0}^{\infty} d_j N_{j,2k}(x)$$

and by the properties of B-splines

$$s(x)=\begin{cases}0, & x\leqslant t_0,\\ q(x), & x\geqslant t_{2k-1},\end{cases}$$

and has possible jump discontinuities in its $(2k-1)^{st}$ derivative at t_0, $t_1, ..., t_{2k-1}$. We note that

$$x_+^{2k-1}=\frac{x^{2k-1}+|x|^{2k-1}}{2}$$

has a jump discontinuity of magnitude $(2k-1)!$ in its $(2k-1)^{st}$ derivative at $x=0$ and none elsewhere. Hence,

$$s(x)=\sum_{l=0}^{2k-1}\frac{\lambda_l}{2}\{(x-t_l)^{2k-1}+|x-t_l|^{2k-1}\},$$

for some constants $\lambda_0, ..., \lambda_{2k-1}$. This can be rewritten as

$$s(x)=\left\{\sum_{l=0}^{2k-1}\frac{\lambda_l}{2}(x-t_l)^{2k-1}\right\}+\left\{\sum_{l=0}^{2k-1}\frac{\lambda_l}{2}|x-t_l|^{2k-1}\right\}.$$

But for $x>t_{2k-1}$ the terms in curly brackets are equal and sum to $q(x)$. Hence the first term equals $q(x)/2$ for all $x>t_{2k-1}$, and since it is a polynomial the equality holds for all $x\in\mathbf{R}$. Thus

$$s(x)=\frac{q(x)}{2}+\left\{\sum_{l=0}^{2k-1}\frac{\lambda_l}{2}|x-t_l|^{2k-1}\right\}.$$

Comparing these last two expressions for $s(x)$ we find

$$\sum_{l=0}^{2k-1} \lambda_l(\cdot - t_l)^{2k-1} = q.$$

Since $\{(\cdot - t_l)^{2k-1} : l=0, ..., 2k-1\}$ forms a basis for π_{2k-1} it follows that the coefficients $\lambda_0, ..., \lambda_{2k-1}$ are uniquely determined by this last equation. This establishes the theorem when $c=0$.

We now turn to the case $c>0$. From Corollary 5, Lemma 1 and Lemma 2

$$q(x) = \sum_{j=-\infty}^{\infty} d_j N_{j,2k}(x) \tag{4.1}$$

is the unique polynomial in π_{2k-1} such that

$$p = \sum_{j=-\infty}^{\infty} d_j \psi_{j,2k} = kA(k)\, c^{2k} \left\{ \sum_{j=-\infty}^{\infty} d_j N_{j,2k} * (\cdot + c^2)^{-(2k+1)/2} \right\}, \tag{4.2}$$

with the last equality holding term by term. Hence

$$\begin{aligned} s &= \sum_{j=0}^{\infty} d_j \psi_{j,2k} \\ &= kA(k)\, c^{2k} \sum_{j=0}^{\infty} d_j \left\{ N_{j,2k} * (\cdot + c^2)^{-(2k+1)/2} \right\} \\ &= kA(k)\, c^{2k} \left\{ \frac{q(\cdot)}{2} + \frac{1}{2} \sum_{l=0}^{2k-1} \lambda_l\, |\cdot - t_l|^{2k-1} \right\} * (\cdot + c^2)^{-(2k+1)/2} \\ &= \frac{p(\cdot)}{2} + \frac{1}{2} \sum_{l=0}^{2k-1} \lambda_l \phi_{l,2k}(\cdot), \end{aligned}$$

where in the second to last equality we have used the already proven result for $c=0$. The last equality follows from (4.1), (4.2) and Lemma 7. ∎

Note that in the special case

$$q = (\cdot - t_0)^{2k-1} = \sum_{j=-\infty}^{\infty} d_j N_{j,2k},$$

Theorem 8 gives the especially simple expression

$$\sum_{j=0}^{\infty} d_j N_{j,2k} = \tfrac{1}{2}\{(\cdot - t_0)^{2k-1} + |\cdot - t_0|^{2k-1}\} = (\cdot - t_0)_+^{2k-1}.$$

5. APPROXIMATION BY ψ-SPLINES

In this section we consider approximation properties of ψ-splines. We use quasi-interpolants to show Jackson-type error estimates for non-uniform meshes and continuous or continuously differentiable functions. The results are generalisations of some of the results of Buhmann [3, 4] for bi-infinite uniform meshes, and of results of Beatson and Powell [2] for quasi-interpolation on a finite mesh with ordinary multiquadrics.

THEOREM 9. *Let* $k \geqslant 1$, $c > 0$ *and mesh* $\mathbf{t}: \cdots < t_{j-1} < t_j < t_{j+1} < \cdots$, *with* $t_{\pm j} \to \pm\infty$ *as* $j \to \infty$ *be given. Suppose the mesh size* $h = \sup_j (t_{j+1} - t_j)$ *is finite. Then for each function* f, *uniformly continuous on* $\mathbf{R}$, *the quasi-interpolant* $\mathcal{L}_{\mathcal{B}} f = \sum_{j=-\infty}^{\infty} f(t_j^*) \psi_{j,2k}$ *satisfies*

$$\| f - \mathcal{L}_{\mathcal{B}} f \|_{L^\infty(\mathbf{R})} \leqslant \left(k + 1 + \frac{c}{h} \right) \omega(f, h).$$

The same result holds when t_j^* *is replaced by* t_{j+k} *in the definition of* $\mathcal{L}_{\mathcal{B}}$.

Proof. Firstly note that

$$| f(x) | \leqslant | f(0) | + (1 + |x|)\, \omega(f, 1), \qquad x \in \mathbf{R}.$$

Hence, $|f(x)|$ grows at most linearly as $x \to \pm\infty$, and $\mathcal{L}_{\mathcal{B}} f$ is well defined by Lemma 2.

From the partition of unity property of the $\psi_{j,2k}$'s

$$f(x) - (\mathcal{L}_{\mathcal{B}} f)(x) = \sum_{j=-\infty}^{\infty} \{ f(x) - f(t_j^*) \}\, \psi_{j,2k}(x)$$

From the properties of the modulus of continuity

$$| f(x) - f(t_j^*) | \leqslant \omega(f, |x - t_j^*|) \leqslant \left(1 + \frac{|x - t_j^*|}{h} \right) \omega(f, h).$$

Hence using also Lemma 2

$$\left| f(x) - \sum_{j=-\infty}^{\infty} f(t_j^*)\, \psi_{j,2k}(x) \right|$$

$$\leqslant k A(k) c^{2k} \int_{-\infty}^{\infty} \frac{\sum_{j=-\infty}^{\infty} | f(x) - f(t_j^*) |\, N_{j,2k}(u)}{((x-u)^2 + c^2)^{(2k+1)/2}}\, du$$

$$\leqslant k A(k) c^{2k}\, \omega(f, h) \int_{-\infty}^{\infty} \frac{\sum_{j=-\infty}^{\infty} \left(1 + \frac{|x - t_j^*|}{h} \right) N_{j,2k}(u)}{((x-u)^2 + c^2)^{(2k+1)/2}}\, du. \qquad (5.1)$$

Now recall that

$$t_j^* = \frac{t_{j+1} + \cdots + t_{j+2k-1}}{2k-1},$$

so that t_j^* is increasing in each of $t_{j+1}, ..., t_{j+2k-1}$. Hence $\max\{t_j^* - t_j, t_{j+2k} - t_j^*\}$ occurs when all the points are as far apart as possible and is kh. Since $\text{supp}(N_{j,2k}) = [t_j, t_{j+2k}]$ it follows that $N_{j,2k}(u)$ is non-zero only when $|u - t_j^*| \leqslant kh$. When this is the case $|x - t_j^*| \leqslant |x-u| + kh$. Substituting into (5.1)

$$|f(x) - (\mathscr{L}_{\mathscr{B}} f)(x)| \leqslant kA(k)\, c^{2k}\, \omega(f, h) \int_{-\infty}^{\infty} \frac{(k+1) + \dfrac{|x-u|}{h}}{((x-u)^2 + c^2)^{(2k+1)/2}}\, du.$$

Using the values for the integrals given in (2.9) and (2.12) we find

$$|f(x) - (\mathscr{L}_{\mathscr{B}} f)(x)| \leqslant \left(k + 1 + \frac{c}{h}\right) \omega(f, h).$$

The argument when we replace t_j^* by t_{j+k} is almost identical. ▮

Corollary 10. *Let $k \in N$ and mesh $\mathbf{t}: t_0 < t_1 < \cdots < t_n$ be given. Let*

$$\mathscr{B} = \text{span}\{1, \phi_{0,2k}, \phi_{1,2k}, ..., \phi_{n,2k}\}.$$

Then

$$\text{dist}(f, \mathscr{B}; L^\infty[t_0, t_n]) \leqslant \left(2k + \frac{c}{h}\right) \omega(f, h)$$

for all $f \in C[t_0, t_n]$, where $h = \max_{0 \leqslant j \leqslant n-1}(t_{j+1} - t_j)$ is the mesh size.

Proof. The proof is divided into two cases.

Case 1. $n \leqslant 2k$. In this case approximate f by the constant function $s(x) = f(t_{[n/2]})$ and note

$$\|f - s\|_{L^\infty[t_0, t_n]} \leqslant (n - [n/2])\, \omega(f, h) \leqslant 2k\omega(f, h).$$

Case 2. $n > 2k$. In this case extend the mesh to $\pm\infty$ by requiring $t_{j+1} - t_j = h$ for all $j \in \mathbf{Z} \setminus [0, n)$. Then set

$$g(x) = \begin{cases} f(t_{k-1}), & x \leqslant t_{k-1}, \\ f(x), & t_{k-1} \leqslant x \leqslant t_{n-k+1}, \\ f(t_{n-k+1}), & t_{n-k+1} \leqslant x. \end{cases}$$

Note that $\max\{t_{k-1} - t_0, t_n - t_{n-k+1}\} \leqslant (k-1)h$ implying that $\|f - g\|_{L^\infty[t_0, t_n]} \leqslant (k-1)\,\omega(f, h)$, and also that g is uniformly continuous on $\mathbf{R}$ with $\omega(g, h) \leqslant \omega(f, h)$. Hence by Theorem 9

$$\left\| g - \sum_{j=-\infty}^{\infty} g(t_{j+k}) \psi_{j, 2k} \right\|_{L^\infty(\mathbf{R})} \leqslant \left(k + 1 + \frac{c}{h} \right) \omega(f, h).$$

Thus

$$\left\| f - \sum_{j=-\infty}^{\infty} g(t_{j+k}) \psi_{j, 2k} \right\|_{L^\infty[t_0, t_n]} \leqslant \left(2k + \frac{c}{h} \right) \omega(f, h). \tag{5.2}$$

Then by Lemma 3 and Theorem 8

$$\sum_{j=-\infty}^{-1} g(t_{j+k})\, \psi_{j, 2k} = f(t_{k-1}) \sum_{j=-\infty}^{-1} \psi_{j, 2k}$$

is in $\operatorname{span}\{1, \phi_{0, 2k}, \phi_{1, 2k}, ..., \phi_{2k-1, 2k}\}$ and hence is in the space $\mathscr{B}$. Similarly

$$\sum_{j=n-2k+1}^{\infty} g(t_{j+k}) \psi_{j, 2k} = f(t_{n-k+1}) \sum_{j=n-2k+1}^{\infty} \psi_{j, 2k}$$

also belongs to the space $\mathscr{B}$. Since $\psi_{j, 2k} \in \mathscr{B}$ for $j = 0, ..., n - 2k$ the corollary follows from (5.2). ∎

THEOREM 11. *Let $k \in \mathbf{N}$ and $k \geqslant 2$. There exists a constant M, depending only on k, with the following property. Let $c > 0$ and mesh $\mathbf{t}: \cdots < t_{j-1} < t_j < t_{j+1} < \cdots$ with $t_{\pm j} \to \pm\infty$ as $j \to \infty$ be given. Suppose the mesh size $h = \sup_j (t_{j+1} - t_j)$ is finite. Then for each function f with f' uniformly continuous on $\mathbf{R}$, the quasi-interpolant,*

$$\mathscr{L}_{\mathscr{B}} f = \sum_{j=-\infty}^{\infty} f(t_j^*) \psi_{j, 2k},$$

satisfies

$$\| f - \mathcal{L}_{\mathcal{B}} f \|_{L^\infty(\mathbf{R})} \leqslant M \left\{ \frac{c^2}{h} + c + h \right\} \omega(f', h).$$

Proof. Firstly note that

$$|f'(x)| \leqslant |f'(0)| + (1 + |x|)\, \omega(f', 1) \qquad x \in \mathbf{R},$$

so that $|f(x)|$ grows at most quadratically as $x \to \pm\infty$. Hence $\mathcal{L}_{\mathcal{B}} f$ is well defined by Lemma 2.

Now from Lemma 3, $\mathcal{L}_{\mathcal{B}}$ reproduces linears. It is after all the analogue of the variation diminishing spline. Hence if p is the linear Taylor polynomial of f at x

$$\sum_{j=-\infty}^{\infty} p(t_j^*)\, \psi_{j,2k}(x) = p(x) = f(x).$$

Thus the approximation error is

$$|f(x) - (\mathcal{L}_{\mathcal{B}} f)(x)| = \left| \sum_{j=-\infty}^{\infty} \{ p(t_j^*) - f(t_j^*) \}\, \psi_{j,2k}(x) \right|.$$

Using the bound

$$|f(t_j^*) - p(t_j^*)| \leqslant |t_j^* - x|\, \omega(f', |t_j^* - x|)$$

from Taylor's theorem the approximation error is bounded above by

$$\sum_{j=-\infty}^{\infty} |x - t_j^*|\, \omega(f', |x - t_j^*|)\, \psi_{j,2k}(x)$$
$$\leqslant \sum_{j=-\infty}^{\infty} |x - t_j^*| \left(1 + \frac{|x - t_j^*|}{h} \right) \omega(f', h)\, \psi_{j,2k}(x). \tag{5.3}$$

Writing

$$S_{2k}(u) = kA(k)\, c^{2k} (u^2 + c^2)^{-(2k+1)/2},$$

as before, the right hand side of (5.3) becomes

$$\omega(f', h) \int_{s=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left\{ \frac{(x - t_j^*)^2}{h} + |x - t_j^*| \right\} N_{j,2k}(u)\, S_{2k}(x-u)\, du. \tag{5.4}$$

But if $N_{j,2k}(u)$ is non-zero then $|u - t_j^*| \leqslant kh$ implying

$$|x - t_j^*| \leqslant |x - u| + kh.$$

Hence (5.4) is bounded above by

$$\omega(f', h) \int_{u=-\infty}^{\infty} \left\{ \frac{1}{h}(x-u)^2 + (2k+1)\,|x-u| + k(k+1)h \right\} S_{2k}(x-u)\, du.$$

Since $k \geqslant 2$, (2.9), (2.12) and (2.21) show there exists a constant M_1, depending only on k such that

$$\int_{-\infty}^{\infty} |u|^j\, S_{2k}(u)\, du \leqslant M_1 c^j, \qquad 0 \leqslant j \leqslant 2,$$

and the result follows. ∎

COROLLARY 12. *Let $k \in \mathbf{N}$ and $k \geqslant 2$. There exists a constant M depending only on k with the following property. Let $c > 0$ and mesh $\mathbf{t} : t_0 < t_1 < \cdots < t_n$ be given. Let*

$$\mathscr{C} = \operatorname{span}\{1, x, \phi_{0,2k}, \phi_{1,2k}, ..., \phi_{n,2k}\}.$$

Then

$$\operatorname{dist}(f, \mathscr{C}; L^\infty[t_0, t_n]) \leqslant M \left\{ \frac{c^2}{h} + c + h \right\} \omega(f', h)$$

for all $f \in C^1[t_0, t_n]$ where $h = \max_{0 \leqslant j < n-1}(t_{j+1} - t_j)$ is the mesh size.

Proof. The proof is divided into two cases.

Case 1. $n \leqslant 2k$. In this case approximate f by the linear function $s(x) = f(t_{[n/2]}) + f'(t_{[n/2]})(x - t_{[n/2]})$ and note

$$\|f - s\|_{L^\infty[t_0, t_n]} \leqslant (n - [n/2])\, h\omega(f', kh) \leqslant k^2 h\omega(f', h).$$

Case 2. $n > 2k$. In this case extend the mesh to $\pm\infty$ by requiring $t_{j+1} - t_j = h$ for all $j \in \mathbf{Z} \backslash [0, n)$. Then set

$$g(x) = \begin{cases} f(t^*_{-1}) + f'(t^*_{-1})(x - t^*_{-1}), & x \leqslant t^*_{-1}, \\ f(x), & t^*_{-1} \leqslant x \leqslant t^*_{n-2k+1}, \\ f(t^*_{n-2k+1}) + f'(t^*_{n-2k+1})(x - t^*_{n-2k+1}), & t^*_{n-2k+1} \leqslant x. \end{cases}$$

Then $\|f - g\|_{L^\infty[t_0, t_n]} \leqslant (k-1)\, h\omega(f', (k-1)h) \leqslant (k-1)^2\, h\omega(f', h)$ and g' is uniformly continuous on $\mathbf{R}$ with $\omega(g', h) \leqslant \omega(f', h)$. By an argument analogous to that in the latter part of the proof of Corollary 10, excepting that the application of Theorem 9 is replaced by an application of Theorem 11, we find that $(\mathscr{L}_{\mathscr{C}}\, g) := \sum_{j=-\infty}^{\infty} g(t_j^*)\, \psi_{j,2k} \in \mathscr{C}$, and

$$\|f - \mathscr{L}_{\mathscr{C}}\, g\|_{L^\infty[t_0, t_n]} \leqslant M_2 \left\{ \frac{c^2}{h} + c + h \right\} \omega(f', h). \quad \blacksquare$$

We now turn to the case $k=1$ discussed in Beatson and Powell [2]. Extend $f \in C^1[t_0, t_n]$ outside $[t_0, t_n]$ by appending first degree Taylor polynomials at t_0 and t_n. They show that the operator $\mathscr{L}_B f$ of Theorem 9 applied to this extended f becomes (in the notation of the current paper)

$$\begin{aligned}
(\mathscr{L}_{\mathscr{B}}\, f)(x) &= \sum_{j=-\infty}^{\infty} f(t_j^*)\, \psi_{j,2}(x) = \sum_{j=-\infty}^{\infty} f(t_{j+1})\, \psi_{j,2}(x) \\
&= \frac{f'(t_0)}{2} \left[(x - t_0) - \phi_0(x) \right] + \frac{f(t_0)}{2} \left[1 + \frac{\phi_1(x) - \phi_0(x)}{t_1 - t_0} \right] \\
&\quad + \sum_{j=1}^{n-1} f(t_j)\, \psi_{j-1,2}(x) \\
&\quad + \frac{f(t_n)}{2} \left[1 - \frac{\phi_n(x) - \phi_{n-1}(x)}{t_n - t_{n-1}} \right] + \frac{f'(t_n)}{2} \left[\phi_n(x) - (t_n - x) \right].
\end{aligned} \tag{5.5}$$

Note that in [2], ψ_j denotes a combination of $\phi_{j-1,2}$, $\phi_{j,2}$, and $\phi_{j+1,2}$ whereas here it denotes a combination of $\phi_{j,2}$, $\phi_{j+1,2}$ and $\phi_{j+2,2}$. They obtain an estimate for $\|f - \mathscr{L}_B f\|$ when f has a Lipschitz derivative. It is natural therefore to seek an estimate in terms of $\omega(f', h)$.

THEOREM 13. *Let $k=1$. There exists a constant M with the following property. Let a mesh $\mathbf{t} : t_0 < t_1 < \cdots < t_n$ be given and $(\mathscr{L}_{\mathscr{B}}\, f)$ be defined by* (5.5), *then*

$$\|f - \mathscr{L}_B f\|_{L^\infty[t_0, t_n]} \leqslant M \left\{ c + h + \frac{c^2}{h} + \frac{c^2}{h} \log\left(1 + \left(\frac{t_n - t_0}{c} \right) \right) \right\} \omega(f', h)$$

for all $f \in C^1[t_0, t_n]$ where h is the mesh size.

Proof. This proof is quite intricate but involves no essentially new ideas. It has therefore been omitted. $\blacksquare$

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